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On Stability of Nonlinear Differential System Via Cone-Perturbing Laipunov Function Method

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Abstract

$Totally\ equistable, totally\ \phi$o – equistable, practically − equistable, practically ϕ_o – equistable of system of differential equations are studied, Cone valued perturbing Liapunov functions method and comparison methods are our technique , Some results of these properties are given.

Keywords: $Totally\ equistable, totally\ \phi$o – equistable, practically − equistable, practically ϕ_o – equistable- Cone valued perturbing Liapunov functions method.

1. Introduction

Consider the non linear system of ordinary differential equations

$$x' = f(t,x), \qquad x(t_0) = x_0 \qquad (1.1)$$

and the perturbed system

$$x' = f(t,x) + R(t,x), \qquad x(t_0) = x_0. \qquad (1.2)$$

Let R^n be Euclidean n –dimensional real space with any convenient norm $\|.\|$, and scalar product $(.,.) \leq \|.\|\|.\|$ .Let for some $\rho > 0$

$$S_\rho = \{x \in R^n, \|x\| < \rho\}.$$

where $f, R \in C[J \times S_\rho, R^n], J = [0,\infty)\ and\ C[J \times S_\rho, R^n]$ denotes the space of continuous mappings $J \times S_\rho$ into R^n .

Consider the scalar differential equations with an initial condition

$$u' = g_1(t,u) \qquad u(t_0) = u_0, \qquad (1.3)$$

$$\omega' = g_2(t,\omega) \qquad \omega(t_0) = \omega_0 \qquad (1.4)$$

and the perturbing equations

$$u' = g_1(t.u) + \varphi_1 \qquad u(t_0) = u_0 \qquad (1.5)$$

$$\omega' = g_2(t.\omega) + \varphi_2 \qquad \omega(t_0) = \omega_0 \qquad (1.6)$$

where $g_1, g_2 \in C\,[J \times R, R]$, $\varphi_1, \varphi_2 \in C\,[J, R]$ respectively.

The following definitions [1] will be needed in the sequal .

Definition 1.1

A proper subset $K\ of\ R^n$ is called a cone if

$$(i)\ \lambda K \subset K, \lambda \geq 0.\ \ (ii) K + K \subset K, (iii) \overline{K} = K, (iv)\ K^0 \neq \emptyset,\ \ (v) K \cap (-K) = \{0\}.$$

where $K\ and\ K^0$ denotes the closure and interior of K respectively and ∂K denote the boundary of K.

Definition 1.2

The set $K^* = \{\phi \in R^n, (\phi, x) \geq 0, x \in K\}$ is called the adjoint cone if it satisfies the properties of the definition 3.1.

$$x \in \partial K\ if\ (\phi, x) = 0\ for\ some\ \phi \in K_0^*\ , K_0 = K/\{0\}.$$

Definition 1.3

A function $g: D \to K, D \subset R^n$ is called quasimonotone relative to the cone K if $x, y \in D$, $y - x \in \partial K$ then there exists $\phi_0 \in K_0^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) > 0$.

Definition 1.4

A function $a(.)$ is said to belong to the class $\mathcal{K}\ if$ $a \in [R^+, R^+]$, $a(0) = 0\ and\ a(r)$ is strictly monotone increasing in r .

2. Totally equistable

In this section we discuss the concept of totally equistable of the zero solution of (1.1) using perturbing Liapuniv functions method and Comparison principle method.

We define for $V \in C[J \times S_\rho, R^n]$, the function $D^+V(t,x)$ by

$$D^+V(t,x)_{1.2} = \lim_{h\to 0} \sup \frac{1}{h}(V(t+h, x + h(f(t,x) + R(t,x)) - V(t,x)).$$

The following definition [7] will be needed in the sequal.

Definition 2.1

The zero solution of the system (1.1) is said to be $T_1 -$ totally equistable (stable with respect to permanent perturbations) , if for every $\epsilon > 0$, $t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0\ and\ \delta_2 = \delta_2(\epsilon) > 0$ such that for every solution of perturbed equation (1.2), the inequality

$$\| x(t, t_0, x_0)\| < \epsilon \ \ for\ t \geq t_0$$

holds ,provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.

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Definition 2.2

The zero solution of the equation (1.3) is said to be $T_1 -$ totally equistable (stable with respect to permanent perturbations) , if for every $\epsilon > 0$, $t_0 \in J$, there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution of perturbed equation (1.5).the inequality

$$u(t, t_0, u_0) < \epsilon, \quad t \geq t_0$$

holds , provided that $u_0 < \delta_1^*$ and $\varphi_1(t) < \delta_2^*$.

Theorem 2.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$ with $g_1(t,0) = g_2(t,0) = 0$ and there exist two Liapunov functions $V_1 \in C[J \times S_\rho, R^n]$ and $V_{2\eta} \in C[J \times S_\rho \cap S_\eta^C, R^n]$ with $V_1(t,0) = V_{2\eta}(t,0) = 0$ where $S_\eta = \{x \in R^n, \|x\| < \eta\}$ for $\eta > 0$ and S_η^C denotes the complement of S_η satisfying the following conditions:

(H_1) $V_1(t,x)$ is locally Lipschitzian in x .

$D^+V_1(t,x) \leq g_1(t, V_1(t,x)) \quad \forall (t,x) \in J \times S_\rho$.

(H_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x

$b(\|x\|) \leq V_{2\eta}(t,x) \leq a(\|x\|) \quad \forall (t,x) \in J \times S_\rho \cap S_\eta^C$.

where $a, b \in \mathcal{K}$ are increasing functions.

(H_3)

$$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \leq g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) \quad \forall (t,x) \in J \times S_\rho \cap S_\eta^C.$$

(H_4) If the zero solution of (1.3) is equistable , and the zero solution of (1.4)is totally equistable .

Then the zero solution of (1.1) is totally equistable.

Proof

Since the zero solution of the system (1.4) is totally equistable , given $b(\epsilon) > 0$, there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0$ and $\delta_2^* = \delta_2^*(\epsilon) > 0$ such that for every solution $\omega(t, t_0, \omega_0)$ of perturbed equation (1.6) the inequality

$$\omega(t, t_0, \omega_0) < \epsilon, \quad t \geq t_0 \tag{2.1}$$

holds , provided that $\omega_0 < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Since the zero solution of (1.3) is equistable given $\frac{\delta_0(\epsilon)}{2}$ and $t_0 \in J$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$u(t, t_0, u_0) < \frac{\delta_0(\epsilon)}{2} \tag{2.2}$$

holds ,provided that $u_0 \leq \delta$

From the condition (H_2) we can find $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^* \tag{2.3}$$

To show that the zero solution of (1.1) is $T_1 -$ totally equistable , it must show that for every $\epsilon > 0$, $t_0 \in J$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ and $\delta_2 = \delta_2(\epsilon) > 0$ such that for every solution $x(t, t_0, x_0)$ of perturbed equation (1.2).the inequality

$$\| x(t, t_0, x_0)\| < \epsilon \quad for\ t \geq t_0$$

holds ,provided that $\|x_0\| < \delta_1$ and $\|R(t,x)\| < \delta_2$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $t_1 > t_0$ such that

$$\|x(t_0, t_0, x_0)\| = \delta_1 \ , \ \|x(t_1, t_0, x_0)\| = \epsilon \quad (2.4) \tag{2.4}$$

$$\delta_1 \leq \|x(t, t_0, x_0)\| \leq \epsilon \qquad for \quad t \in [t_0, t_1].$$

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$

Since $V_1(t,x)$ and $V_{2\eta}(t,x)$ are Lipschitzian in x for constants M_1 and M_2 respectively.

Then

$$D^+V_1(t,x)_{1.2} + D^+V_{2\eta}(t,x)_{1.2} \leq D^+V_1(t,x)_{1.1} + D^+V_{2\eta}(t,x)_{1.1} + M\|R(t,x)\|$$

where $M = M_1 + M_2$ From the condition (H_3) we obtain the differential inequality

$$D^+V_1(t,x) + D^+V_{2\eta}(t,x) \leq g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) + M\|R(t,x)\|$$

$for \quad t \in [t_0, t_1]$Then we have

$$D^+m(t,x) \leq g_2(t, m(t,x)) + M\|R(t,x)\|$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$

Applying the comparison Theorem (1.4.1) of [7] , it yields

$$m(t,x) \leq r_2(t, t_0, \omega_0) \qquad for \qquad t \in [t_0, t_1].$$

where $r_2(t, t_0, \omega_0)$ is the maximal solution of the perturbed equation (1.6)

Define $\varphi_2(t) = M\|R(t,x)\|$

To prove that

$$r_2(t, t_0, \omega_0) < b(\epsilon).$$

It must be show that

$\omega_0 < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$.

Choose $u_0 = V_1(t_0, x_0)$. From the condition (H_1) and applying the comparison Theorem of [7], it yields

$$V_1(t,x) \leq r_1(t, t_0, u_0)$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From (2.2) at $t = t_0$

$$V_1(t_0, x_0) \leq r_1(t_0, t_0, u_0) < \frac{\delta_0(\epsilon)}{2} \tag{2.5}$$

From the condition (H_2) and (2.4) , at $t = t_0$

$$V_{2\eta}(t_0, x_0) \le a(\|x_0\|) \le a(\delta_1) \quad (2.6)$$

From (2.3), we get

$$\omega_0 = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0) \le \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1^*.$$

Since $\varphi_2(t) = M\|R(t,x)\| \le M\delta_2 = \delta_2^*$

From (2.1) ,we get

$$m(t,x) \le r_2(t, t_0, \omega_0) < b(\epsilon) \quad (2.7)$$

Then from the condition (H_2), (2.4) and (2.7) we get $t = t_1$

$$b(\epsilon) = b(\|x(t_1)\|) \le V_{2\eta}(t_1, x(t_1)) < m(t_1, x(t_1)) \le r_2(t_1, t_0, \omega_0) < b(\epsilon).$$

This is acontradiction ,then it must be $\| x(t, t_0, x_0)\| < \epsilon \quad for\ t \ge t_0$ holds ,provided that $\|x_0\| < \delta_1$and $\|R(t,x)\| < \delta_2$.

Therefore the zero solution of (1.1) is totally equistable.

3. Totally ϕ_0 −equistable.

In this section we discuss the concept of Totally ϕ_0 −equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definition [3] will be needed in the sequal.

Definition 3.1

The zero solution of the system (1.1) is said to be totally ϕ_0 −equistable (ϕ_0 −equistable with respect to permanent perturbations) ,if for every $\epsilon > 0$, $t_0 \in J$ and $\phi_0 \in K_0^*$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0\ and\ \delta_2 = \delta_2(\epsilon) > 0$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < \epsilon \quad for\ t \ge t_0$$

holds ,provided that $(\phi_0, x_0) < \delta_1$and $\|R(t,x)\| < \delta_2$where $x(t, t_0, x_0)$ is the maximal solution of perturbed equation (1.2).

Let for some $\rho > 0$

$$S_\rho^* = \{x \in R^n, (\phi_0, x) < \rho, \phi_0 \in K_0^*\}$$

Theorem 3.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$with $g_1(t,0) = g_2(t,0) = 0$ and let there exist two cone valued Liapunov functions $V_1 \in C[J \times S_\rho^*, K]$ and $V_{2\eta} \in C[J \times S_\rho^* \cap S_\eta^{*C}, K]$ with $V_1(t,0) = V_{2\eta}(t,0) = 0$ where$S_\eta^* = \{x \in K, (\phi_0, x) < \eta\ , \phi_0 \in K_0^*\}\ \ for\ \eta > 0\ and\ S_\eta^{*C}$ denotes the complement of S_η^* satisfying the following conditions:

(h_1) $V_1(t,x)$ is locally Lipschitzian in x and $D^+(\phi_0, V_1(t,x)) \le g_1(t, V_1(t,x))$ for $(t,x) \in J \times S_\rho^*$.

(h_2) $V_{2\eta}(t,x)$ is locally Lipschitzian in x and $b(\phi_0, x) \le (\phi_0, V_{2\eta}(t,x)) \le a(\phi_0, x)$ for $(t, x_t) \in J \times S_\rho^* \cap S_\eta^{*C}$.
where $a, b \in \mathcal{K}$ are increasing functions.

(h_3) $D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2\eta}(t,x)) \le g_2(t, V_1(t,x) + V_{2\eta}(t,x))$ for $(t,x) \in J \times S_\rho^* \cap S_\eta^{*C}$.

(h_4) If the zero solution of (1.3) is $\phi_0 -$ equistable , and the zero solution of (1.4)is totally $\phi_0 -$ equistable . then the zero solution of (1.1) is totally $\phi_0 -$ equistable.

Proof

Since the zero solution of (1.4) is totally $\phi_0 -$ equistable , given , given $b(\epsilon) > 0$ there exist two positive numbers $\delta_1^* = \delta_1^*(\epsilon) > 0\ and \delta_2^* = \delta_2^*(\epsilon) > 0$ such that the inequality

$$(\phi_0, r_2(t, t_0, \omega_0)) < \epsilon, \quad t \ge t_0 \quad (3.1)$$

holds , provided that $(\phi_0, \omega_0) < \delta_1^*$ and $\varphi_2(t) < \delta_2^*$. where $r_2(t, t_0, \omega_0)$ is the maximal solution of perturbed equation (1.6).

Since the zero solution of the system (1.3) is $\phi_0 -$ equistable , given $\frac{\delta_0(\epsilon)}{2}\ and\ t_0 \in J$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$(\phi_0, r_1(t, t_0, u_0)) < \frac{\delta_0(\epsilon)}{2} \quad (3.2)$$

holds ,provided that $(\phi_0, u_0) \le \delta$ where $r_1(t, t_0, u_0)$ is the maximal solution of(1.3)

From the condition (h_2) we can choose $\delta_1 = \delta_1(\epsilon) > 0$ such that

$$a(\delta_1) + \frac{\delta_0}{2} < \delta_1^* \quad (3.3)$$

To show that the zero solution of (1.1) is $T_1 -$ totally$\phi_0 -$ equistable ,it must be prove that for every $\epsilon > 0$, $t_0 \in J$ and $\phi_0 \in K_0^*$ there exist two positive numbers $\delta_1 = \delta_1(\epsilon) > 0$ $and\ \delta_2 = \delta_2(\epsilon) > 0$ such that the inequality $(\phi_0, x(t, t_0, x_0)) < \epsilon \quad for\ t \ge t_0$
holds ,provided that $(\phi_0, x_0) < \delta_1$and $\|R(t,x)\| < \delta_2$ where $x(t, t_0, x_0)$ is the maximal solution of perturbed equation (1.2).

Suppose that is false, then there exists a solution $x(t, t_0, x_0)$ of (1.2) with $t_1 > t_0$ such that

$$(\phi_0, x(t_0, t_0, x_0)) = \delta_1\ ,\ (\phi_0, x(t_1, t_0, x_0)) = \epsilon \quad (3.4)$$

$$\delta_1 \le (\phi_0, x(t, t_0, x_0)) \le \epsilon \quad for\ \ t \in [t_0, t_1].$$

Let $\delta_1 = \eta$ and setting $m(t,x) = V_1(t,x) + V_{2\eta}(t,x)$

Since $V_1(t,x)\ \ and\ \ V_{2\eta}(t,x)$ are Lipschitzian in x for constants $M_1\ and\ M_2$ respectively.

Then

$$D^+(\phi_0, V_1(t,x))_{1.2} + D^+(\phi_0, V_{2\eta}(t,x))_{1.2} \le D^+(\phi_0, V_1(t,x))_{1.1} + D^+(\phi_0, V_{2\eta}(t,x))_{1.1} + M\|R(t,x)\|$$

where $M = M_1 + M_2$ From the condition (h_3) we obtain the differential inequality

$$D^+(\phi_0, V_1(t,x)) + D^+(\phi_0, V_{2\eta}(t,x)) \le g_2\left(t, V_1(t,x) + V_{2\eta}(t,x)\right) + M\|R(t,x)\|$$

$for \quad t \in [t_0, t_1]$Then we have

$$D^+(\phi_0, m(t,x)) \le g_2\big(t, \mathrm{m(t,x)}\big) + M\|R(t,x)\|$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2\eta}(t_0, x_0)$

Applying the comparison Theorem of [7] , yields

$$(\phi_0, m(t,x)) \le (\phi_0, r_2(t, t_0, \omega_0)) \quad for \ t \in [t_0, t_1].$$

Define $\varphi_2(t) = M\|R(t,x)\|$

To prove that

$$(\phi_0, r_2(t, t_0, \omega_0)) < b(\epsilon).$$

It must be shown that

$(\phi_0, \omega_0) < \delta_1^*$ and $\varphi_2(\mathrm{t}) < \delta_2^*$.

Choose $u_0 = V_1(t_0, x_0)$. From the condition (h_1) and applying the comparison Theorem [7], it yields

$$(\phi_0, V_1(t,x)) \le (\phi_0, r_1(t, t_0, u_0))$$

From (3.2) at $t = t_0$

$$(\phi_0, V_1(t_0, x_0)) \le (\phi_0, r_1(t_0, t_0, u_0)) < \frac{\delta_0(\epsilon)}{2} \tag{3.5}$$

From the condition (h_2) and (3.4) , at $t = t_0$

$$(\phi_0, V_{2\eta}(t_0, x_0)) \le a(\phi_0, x_0) \le a(\delta_1) \tag{3.6}$$

From (3.3), we get

$$(\phi_0, \omega_0) = (\phi_0, V_1(t_0, x_0)) + (\phi_0, V_{2\eta}(t_0, x_0)) \le \frac{\delta_0(\epsilon)}{2} + a(\delta_1) < \delta_1^*.$$

Since $\varphi_2(t) = M\|R(t,x)\| \le M\delta_2 = \delta_2^*$

From (3.1) ,we get

$$\big(\phi_0, m(t,x)\big) \le \big(\phi_0, r_2(t, t_0, \omega_0)\big) < b(\epsilon) \tag{3.7}$$

Then from the condition (h_2), (3.4) and (3.7) we get at $t = t_1$

$$b(\epsilon) = b(\phi_0, x(t_1)) \le (\phi_0, V_{2\eta}\big(t_1, x(t_1)\big) < (\phi_0, m\big(t_1, x(t_1)\big) \le (\phi_0, r_2(t_1, t_0, \omega_0)) < b(\epsilon).$$

This is a contradiction ,then

$$(\ \phi_0, x(t, t_0, x_0)) < \epsilon \quad for \ t \ge t_0$$

provided that $(\ \phi_0, x_0) < \delta_1$and $\|R(t,x)\| < \delta_2$where $x(t, t_0, x_0)$ is the maximal solution of perturbed equation (1.2).

Therefore the zero solution of (1.1) is totally $\phi_0 -$ equistable.

4. Practically equistable

In this section, we discuss the concept of practically equistable of the zero solution of (1.1) using perturbing Liapunov functions method and Comparison principle method.

The following definition [5] will be needed in the sequal.

Definition 4.1

Let $0 < \lambda < A$ be given . The system (1.1) is said to be practically equistable if for $t_0 \in J$ such that the inequality

$$\|\ x(t, t_0, x_0)\| < A \qquad for\ t \ge t_0 \tag{4.1}$$

holds ,provided that $\|x_0\ \| < \lambda$ where $x(t, t_0, x_0)$ is any solution of (1.1).

In case of uniformly practically equistable ,the inequality (4.1) holds for any t_0.

We define

$$S(A) = \{x \in R^n : \|x\| \le A, \quad A > 0\}.$$

Theorem 4.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$with$g_1(t,0) = g_2(t,0) = 0$and there exist two Liapunov functions $\mathrm{V}_1 \in \mathrm{C}[\mathrm{J} \times \mathrm{S(A)}, \mathrm{R^n}]$ and $\mathrm{V}_{2\eta} \in \mathrm{C}[\mathrm{J} \times \mathrm{S(A)} \cap \mathrm{S(B)}^{\mathrm{C}}, \mathrm{R^n}]$with

$V_1(t,0) = \mathrm{V}_{2\mathrm{B}}(t,0) = 0$

where $S(B) = \{x \in R^n, \|x\| < B\ ,0 < \mathrm{B} < \mathrm{A}\}$ and $\mathrm{S(B)}^C$ denotes the complement of $S(B)$ satisfying the following conditions:

(I) $\mathrm{V_1(t,x)}$ is locally Lipschitzian in x .

$$\mathrm{D^+V_1(t,x)} \le \mathrm{g_1\big(t, V_1(t,x)\big)} \quad \forall \mathrm{(t,x)} \in \mathrm{J} \times \mathrm{S(A)}.$$

(II) $\mathrm{V_{2B}(t,x)}$ is locally Lipschitzian in x .

$$\mathrm{b(\|x\|)} \le \mathrm{V_{2B}(t,x)} \le \mathrm{a(\|x\|)} \quad \forall \mathrm{(t,x)} \in \mathrm{J} \times \mathrm{S(A)} \cap \mathrm{S(B)^C}.$$

where $\mathrm{a, b} \in \mathcal{K}$ are increasing functions.

(III)

$$\mathrm{D^+V_1(t,x)} + \mathrm{D^+V_{2\eta}(t,x)} \le \mathrm{g_2\big(t, V_1(t,x) + V_{2B}(t,x)\big)} \quad \forall \mathrm{(t,x)} \in \mathrm{J} \times \mathrm{S(A)} \cap \mathrm{S(B)^C}.$$

(IV) If the zero solution of (1.3) is equistable , and the zero solution of (1.4)is uniformly practically equistable .

Then the zero solution of (1.1) is practically equistable.

Proof

Since the zero solution of (1.4) is uniformly practically equistable, given $0 < \lambda_0 < A$ such that for every solution $\omega(t, t_0, \omega_0)$ of (1.4) the inequality

$$\omega(t, t_0, \omega_0) < b(A) \tag{4.2}$$

holds provided $\omega_0 \le \lambda_0$.

Since the zero solution of the system (1.3) is equistable , given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$there exist

$\delta = \delta(t_0, \epsilon) > 0$ such that for every solution $u(t, t_0, u_0)$ of (1.3)

$$u(t, t_0, u_0) < \frac{\lambda_0}{2} \tag{4.3}$$

holds provided that $u_0 \leq \delta$.

From the condition (II) we can find $\lambda > 0$ such that

$$a(\lambda) + \frac{\lambda_0}{2} \leq \lambda_0 \tag{4.4}$$

To show that The zero solution of (1.1) practically equistable , it must be exist $0 < \lambda < A$ such that for for any solution $x(t, t_0, x_0)$ of (1.1) the inequality

$$\| x(t, t_0, x_0)\| < A \qquad for\ t \geq t_0$$

holds ,provided that $\|x_0\| < \lambda$.

Suppose that this is false, then there exists a solution $x(t, t_0, x_0)$ of (1.1) with $t_1 > t_0$ such that $\|x(t_0, t_0, x_0)\| = \lambda$, $\|x(t_1, t_0, x_0)\| = A$ (4.5)

$$\lambda \leq \|x(t, t_0, x_0)\| \leq A \qquad for \quad t \in [t_0, t_1].$$

Let $\lambda = B$ and setting

$$m(t, x) = V_1(t, x) + V_{2\eta}(t, x)$$

From the condition (III) we obtain the differential inequality $for \quad t \in [t_0, t_1]$

$$D^+ m(t, x) \leq g_2(t, \mathrm{m(t, x)})$$

Let $\omega_0 = m(t_0, x_0) = V_1(t_0, x_0) + V_{2B}(t_0, x_0)$

Applying the comparison Theorem [7] , yields

$$m(t, x) \leq r_2(t, t_0, \omega_0) \qquad for \quad t \in [t_0, t_1].$$

where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4)

To prove that

$$r_2(t, t_0, \omega_0) < b(A).$$

It must be show that $\omega_0 \leq \lambda_0$.

Choose $u_0 = V_1(t_0, x_0)$, from the condition (II) and applying the comparison Theorem of [7], yields

$$V_1(t, x) \leq r_1(t, t_0, u_0)$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (1.3).

From (4.3) at $t = t_0$

$$V_1(t, x) \leq r_1(t, t_0, u_0) < \frac{\lambda_0}{2}$$

From the condition (II) and (4.5) , at $t = t_0$

$$V_{2B}(t_0, x_0) \leq a(\|x(t_0)\|) \leq a(\lambda)$$

From (4.4),(4.6) and(4.7), we get

$$\omega_0 = V_1(t_0, x_0) + V_{2B}(t_0, x_0) \leq \lambda_0$$

From (4.2) ,we get

$$m(t, x) \leq r_2(t, t_0, \omega_0) < b(A)$$

Then from the condition(II) , (4.5) and (4.8), we get at $t = t_1$

$$b(A) = b(\|x(t_1)\|) \leq V_{2B}(t_1, x_1) < m(t_1, x(t_1)) \leq r_2(t_1, t_0, \omega_0) < b(A).$$

This is a contradiction ,then

$$\| x(t, t_0, x_0)\| < A \qquad for\ t \geq t_0$$

provided that $\|x_0\| < \lambda$.

Therefore the zero solution of (1.1) is practically equistable.

5. practically ϕ_0 −equistable

In this section we discuss the concept of practically ϕ_0 −equistable of the zero solution of (1.1) using cone valued perturbing Liapunov functions method and Comparison principle method.

The following definitions [6] will be needed in the sequal .

Definition 5.1

Let $0 < \lambda < A$ be given . The system (1.1) is said to be practically ϕ_0 −equistable, if for $t_0 \in J$ and $\phi_0 \in K_0^*$ such that the inequality

$$(\phi_0, x(t, t_0, x_0)) < A \qquad for\ t \geq t_0 \tag{5.1}$$

holds ,provided that $(\phi_0, x_0) < \lambda$ where $x(t, t_0, x_0)$ is the maximal solution of (1.1)

In case of uniformly practically $\phi_0 -$ equistable ,the inequality (5.1) holds for any t_0.

We define

$$S^*(\mathrm{A}) = \{\mathrm{x} \in \mathrm{K}, (\phi_0, \mathrm{x}) < \mathrm{A}, \phi_0 \in \mathrm{K}_0^*\}$$

Theorem 5.1

Suppose that there exist two functions $g_1, g_2 \in C[J \times R, R]$with $g_1(t, 0) = g_2(t, 0) = 0$ and let there exist two cone valued Liapunov functions $V_1 \in C[J \times S^*(A), K]$ and $V_{2B} \in C[J \times S^*(A) \cap S^*(B)^C, K]$ with $V_1(t, 0) = V_{2B}(t, 0) = 0$ where $S^*(B) = \{x \in K, (\phi_0, x_0) < B, 0 < B < A,\ \phi_0 \in K_0^*\}$ and $S^*(B)^C$ denotes the complement of $S^*(B)$ satisfying the following conditions:

(i) $V_1(t, x)$ is locally Lipschitzian in x relative to K .

$$D^+(\phi_0, V_1(t, x)) \leq g_1(t, V_1(t, x)) \quad \forall (t, x) \in J \times S^*(\mathrm{A}).$$

(ii) $V_{2B}(t, x)$ is locally Lipschitzian in x relative to K .

$$b(\phi_0, \mathrm{x}) \leq (\phi_0, V_{2B}(t, x)) \leq a(\phi_0, \mathrm{x}) \quad \forall (t, x) \in J \times S^*(\mathrm{A}) \cap S^*(\mathrm{B})^C.$$

where $a, b \in \mathcal{K}$ are increasing functions.

(iii) $D^+(\phi_0, V_1(t, x)) + D^+(\phi_0, V_{2B}(t, x)) \leq g_2(t, V_1(t, x) + V_{2B}(t, x))$

$$\forall (t, x) \in J \times S^*(\mathrm{A}) \cap S^*(\mathrm{B})^C.$$

(iv) If the zero solution of (1.3) is $\phi_0 -$ equistable, and the zero solution of (1.4) is uniformly practically $\phi_0 -$ equistable.

Then the zero solution of (1.1) is practically $\phi_0 -$ equistable.

Proof

Since the zero solution of the system (1.4) is uniformly practically ϕ_0 −equistable, given given

$0 < \lambda_0 < a(B)$ for $a(B) > 0$ such that the inequality

$(\phi_0, r_2(t, t_0, \omega_0)) < a(B)$ (5.2)

holds provided $(\phi_0, \omega_0) \leq \lambda_0$.where $r_2(t, t_0, \omega_0)$ is the maximal solution of (1.4).

Since the zero solution of the system (1.3) is ϕ_0 −equistable , given $\frac{\lambda_0}{2}$ and $t_0 \in R_+$ there exist $\delta = \delta(t_0, \lambda_0)$ such that the inequality

$\left(\phi_0, r_1(t, t_0, u_0)\right) < \frac{\lambda_0}{2}$ (5.3)

From the condition (ii) , assume that

$a(B) \leq b(A)$ (5.4)

also we can choose $\lambda_1 > 0$ such that

$a(\lambda) + \frac{\lambda_0}{2} \leq \lambda_0$ (5.5)

To show that the zero solution of (1.1) is practically $\phi_0 -$ equistable. It must be show that for $0 < \lambda < A$, $t_0 \in J$ *and* $\phi_0 \in K_0^*$ such that the inequality

$(\phi_0, x(t, t_0, x_0)) < A$ *for* $t \geq t_0$

holds ,provided that $(\phi_0, x_0) < \lambda$ where $x(t, t_0, x_0)$ is the maximal solution of (1.1).

Suppose that is false, then there exists a solution $x(t, t_0, x_0)$of (1.1) with $t_2 > t_1 > t_0$ such that for $(\phi_0, x_0) < \lambda$ where $\lambda = \min(\lambda_0, \lambda_1)$

$(\phi_0, x(t_1, t_0, x_0)) = \lambda_1$, $(\phi_0, x(t_2, t_0, x_0)) = A$ (5.6)

$\lambda_1 \leq (\phi_0, x(t, t_0, x_0)) \leq A$ *for* $t \in [t_1, t_2]$

Let $\lambda_1 = B$ and setting

$m(t, x) = V_1(t, x) + V_{2B}(t, x)$

From the condition (iii) we obtain the differential inequality

$D^+(\phi_0, m(t, x)) \leq (\phi_0, g_2(t, m(t, x))$ *for* $t \in [t_1, t_2]$

$\omega_0 = m(t_1, x(t_1)) = V_1(t_1, x(t_1)) + V_{2B}(t_1, x(t_1))$

Applying the comparison Theorem of [7] , yields

$(\phi_0, m(t, x)) \leq (\phi_0, r_2(t, t_0, \omega_0))$

To prove that

$(\phi_0, r_2(t, t_0, \omega_0)) < a(B)$

It must be show that

$(\phi_0, \omega_0) \leq \lambda_0$

Choose $u_0 = V_1(t_0, x_0)$ From the condition (i) and applying the comparison Theorem [7] it yields

$\left(\phi_0, V_1(t, x)\right) \leq \left(\phi_0, r_1(t, t_0, u_0)\right)$

From (5.3) at $t = t_1$

$\left(\phi_0, V_1(t, x)\right) \leq \left(\phi_0, r_1(t, t_0, u_0)\right) < \frac{\lambda_0}{2}$ (5.8)

From the condition (ii) and (5.6) , at $t = t_1$

$\left(\phi_0, V_{2B}(t_1, x(t_1))\right) \leq \left(\phi_0, x(t_1)\right) \leq a(\lambda_1)$ (5.9)

From (5.5),(5.8) and(5.9), we get

$(\phi_0, \omega_0) = (\phi_0, V_1(t_1, x(t_1)) + (\phi_0, V_{2B}(t_1, x(t_1)) \leq \lambda_0$

From (5.2) ,we get

$\left((\phi_0, m(t, x)\right) \leq \left(\phi_0, r_2(t, t_0, \omega_0)\right) < a(B)$ (5.10)

Then from the condition (ii) ,,(5.4) , (5.6) and (5.10), we get at $t = t_2$

$$\begin{aligned} b(A) &= b\left(\phi_0, x(t_2)\right) \\ &\leq \left(\phi_0, m(t_2, x(t_2)\right) \\ &< \left(\phi_0, r_2(t_2, t_0, \omega_0)\right) \\ &< a(B) \\ &\leq a(A). \end{aligned}$$

which leads to a contradiction ,then it must be

$(\phi_0, x(t, t_0, x_0)) < A$ for $t \geq t_0$

holds ,provided that $(\phi_0, x_0) < \lambda$ Therefore the zero solution of (1.1) is practically $\phi_0 -$ equistable.

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