Benha Journal of Applied Sciences (BJAS) *print: ISSN 2356–9751*
Vol. (1) (2016), 96-106 *online: ISSN 2356–976x*
http:// bjas.bu.edu.eg

Periodic Solutions of Modified Duffing Equation Subjected to Abi-Harmonic Parametric and External Excitations

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Abstract

In this paper, we investigated the periodic solutions of modified Duffing's equation subjected to a bi-harmonic parametric and external excitations. The method of multiple scales is used to construct a first order uniform expansion yielding two first-order non-linear ordinary differential equations are derived for the evolution of the amplitude and phase. These oscillations involve a super-harmonic and sub-superharmonic oscillations. Steady state responses and their stability are computed for selected values of the system parameters. The effects of these (quadratic and cubic) non-linearities on these oscillations are specifically investigated. With this study, it has been verified that the qualitative effects of these non-linearities are different. Regions of hardening (softening) behavior of the system are exist for the case of sub-superharmonic oscillation. Numerical solutions are presented which illustrate the behavior of the steady-state response amplitude as a function of the detuning parameter.

Keywords: Weakly non-linear differential equation, MEMS, multiple scale method, parametric excitation and external excitation.

1. Introduction

In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical and engineering problems linear and nonlinear. In the present study, we use the method of multiple scales(MMS) for determination of the response of non-linearly oscillator to external excitation. For a comprehensive review, we refer the reader to [1].

Zavodney et al. [2] studied the response of a model includes quadratic and cubic geometric non-linearities. They found that stable limit cycles can exist. Zavodney and Nayfeh [3] investigated the dynamics of a cantilever beam carrying a lumped mass. They modeled the structure with cubic geometric and intertia non-linearities. A thorough analysis of the governing equation of the motion has provided an accurate model of the dynamic response of such devices [4-6], which has been compared well with experimental results. The method of multiple scales is applied throughout. Asfar [7] took material non-linearity into consideration in the analysis of the performance of an elastomeric damper with a spring harding cubic effects near primary resonance condition applying multiple scale method. Kamel and Amer [8] studied the behavior of one-degree-of-freedom system with different quadratic damping and cubic stiffness non-linearities simulating the axial vibration of a cantilever beam under multi parametric excitation forces.The method of multiple scales has been used to solve the equations to first order perturbation. The theoretical results showed that controlled variations in the softening stiffness can have a significant effect on the overall non-linear response of the system, by making the overall effect hardening, softening, or approximately linear. Eissa and Amer [9] studied the vibration of a second order system to the first mode of a cantilever beam subjected to both external and parametric excitation at primary and sub-harmonic resonance. They analyzed the system using the method of multiple scales. Nayfeh [10] compared application of the method of multiple scales with reconstitution and the generalized method of averaging for determining higher-order approximations of three single-degree-of-freedom systems and a two-degree-of-freedom system. He showed that the second-order frequency-response equation possesses spurious solutions for the case of softening nonlinearity. El-Bassiouny [11] investigated the effects of quadratic and cubic non-linearities in elastomeric material dampers on torsional vibration control. The multiple time scale is used to solve the stability equations at primary resonance. The multiple scale perturbation technique is applied throughout. A threshold value of linear damping has been obtained, where the system vibration can be reduced dramatically. Masana and Daqaq [12] have carried out detailed studies of the post-buckled piezoelectric beam. However, the advantage of the bistable device over the linear device was not uniform, with the exception at very low frequencies when the bistable harvester was excited into high-energy orbits but the linear harvester was weakly excited. Superharmonic dynamics were specifically considered in a series of comparable tests and simulations [13]. Sebald et al [14] described a similar technique whereby an impulsive voltage could be applied in the harvesting circuit to achieve the same objective. theoretically. This reduces the computational cost since the electrostatic force term in the discretized equation will not require complicated numerical integration (integrating a numerator term over a denominator term numerically is computationally expensive) [15].

The problem of parametric resonance arises in many branches of physics and engineering. One of the important problems is that of dynamic instability. There are cases in which the influence of a small vibration loading can stabilize a system which is statically unstable and vice-versa. There are many books devoted to the analysis and applications of the problem of parametric excitation [16]. As an example McLachlan [17] discussed the theory and applications of the Mathieu functions. The treatment of the parametric excitation system, having many degrees of freedom and distinct natural frequencies, is usually operated by the multiple scales method as given by Nayfeh [18]. The interfacial stability with periodic forces is a relatively new topic in the theory of hydrodynamic stability. The mathematical analysis is more difficult because: (a) the method of normal modes is not applicable and (b) the linearized differential equations have time-dependent coefficients so that, the exponential time dependence of the perturbation is not separable. Elhefnawy and El-Bassiouny [19] studied the non-linear stability and chaos in Electrohydrodynamics. El- Bassiouny[20] investigated the principal parametric resonance of a single-degree-of-freedom system with nonlinear two-frequency parametric and self-excitations. Qualitative analysis and asymptotic expansion techniques are employed to predict the existence of steady state responses. Stability condition is investigated. The effect of damping, magnitudes of non-linear excitation and self-excitation are analyzed. El-Bassiouny and Eissa [21] analyzed the behavior of two-degrees-of-freedom vibrating mechanical structure, which is described by two nonlinear differential equations with quadratic and cubic non-linearity's, subjected to multi-frequency parametric excitations in the presence of two-to-one internal resonance. Two approximate methods (the multiple scales and the generalized synchronization) are used to obtain a uniform first-order expansion. The results obtained by the two methods are in excellent agreement. Elnaggar et al.[22] studied harmonic and sub-harmonic resonance of micro-electro mechanical system (MEMS) subjected to a weakly non-linear parametric and external excitation. Elnaggar et al. [23] used the method of multiple scales to investigated the saddle node bifurcation control for an odd non-linearity problem. Elnaggar et al. [24] analyzed the perturbation analysis of an electrostatic Micro-Electro-Mechanical system(MEMS) subjected to external and non-linear parametric excitations. Harmonic, sub-harmonic and super-harmonic resonance of weakly non-linear dynamical system subjected to external excitation, parametric excitation or both are investigated by Elnaggar et al. [25].

In this paper an analysis of super-harmonic oscillation of order two and sub-superharmonic oscillation of order three-to-two are illustrated. Two first-order non-linear ordinary differential equations are derived for the evolution of the amplitude and phase with damping, non-linearity, and all possible solutions based on mathematically justified multiple scales method. Stability analysis are carried out for each case.

2. Formulation of the problem and perturbation analysis

The mathematical model of the Micro-Electro Mechanical Systems (MEMS) is represented by the following weakly non-linear second order differential equation

Equation (1) represent modified Duffing equation subjected to weakly non-linear parametric and external excitations. This equation describes the main motions at time scales of the natural vibrations of the microstructure and fast dynamic at time scales of the high-frequency volotage. Where the dots indicate differentiation with respect to t, ε is a small parameter, μ is the coefficient of viscous damping, ω_0 is the linear natural frequency, Ω is frequency of the external excitation, α is the coefficient of linear and non-linear terms respectively, α_1 and α_2 are the coefficients of the non-linear terms, F_1 and F_2 are the coefficients of linear and nonlinear parametric excitations respectively.

$$u'' + 2\varepsilon\mu u' + \omega_0^2 u + \varepsilon(\alpha_1 u^2 + \alpha_2 u^3) - \varepsilon\alpha(2u + 3u^2 + 4u^3) - \varepsilon(2u + 3u^2 + 4u^3)$$
$$(F_1 \cos[\Omega t] + F_2 \cos[2\Omega t]) - \varepsilon(\alpha + F_1 \cos[\Omega t] + F_2 \cos[2\Omega t]) = 0 \qquad (1)$$

To determine a first-order uniform expansion of the solutions of equation (1), one can use the method of multiple scales [26-29]. Let

$$u(t;\varepsilon)=u_0(T_0,T_1)+\varepsilon u_1(T_0,T_1)+O(\varepsilon^2),\qquad T_n=\varepsilon^n t \tag{2}$$

Where $T_0=t$ is the first scale associated with changes occurring at the frequencies ω_0 and Ω and $T_1=\varepsilon t$ is a slow scale associated with modulations in the amplitude. Denote $D_0=\frac{\partial}{\partial T_0}$ and $D_1=\frac{\partial}{\partial T_1}$.

Substituting from equation (2) into equation (1) and equating the coefficients of like power of ε, one has the following equations to order $O(1)$ and to order $O(\varepsilon)$:

$$D_o^2u_o+\omega_o^2u_o=0 \tag{3}$$

$$\begin{aligned}D_o^2u_1+\omega_o^2u_1= &-2\mu D_ou_o-2D_oD_1u_0+F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o]-\alpha_2u_o^3\\ &+2u_o(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o])+3\alpha u_o^2+2\alpha u_o+\alpha\\ &+3u_o^2(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o])-\alpha_1u_o^2+4\alpha u_o^3\\ &+4u_o^3(F_1\cos[\Omega T_o]+F_2\cos[2\Omega T_o])\end{aligned} \tag{4}$$

The solution of equation (3) can be expression in the form

$$u_0(T_0,T_1)=A(T_1)e^{i\omega_0T_0}+c.c, \tag{5}$$

where A is the amplitude of the response which is a function of T_1 and $c.c$ denotes the complex conjugate, substitute equation (5) into equation (4), we get

$$\begin{aligned}D_o^2u_1+\omega_o^2u_1= &-(-2\alpha A+2i\mu\omega_oA-12\alpha A^2\bar{A}+3A^2\alpha_2\bar{A}+2i\omega_oA')e^{i\omega_oT_o}\\ &+6\alpha A\bar{A}-2A\alpha_1\bar{A}+\frac{3}{2}F_1\bar{A}^2e^{i(\Omega-2\omega_o)T_o}+\frac{3}{2}F_2\bar{A}^2e^{i(2\Omega-2\omega_o)T_o}\\ &+(F_1\bar{A}+6AF_1\bar{A}^2)e^{i(\Omega-\omega_o)T_o}+(F_2\bar{A}+6AF_2\bar{A}^2)e^{i(2\Omega-\omega_o)T_o}\\ &+(\frac{F_1}{2}+3AF_1\bar{A})e^{i\Omega T_o}+(\frac{F_2}{2}+3AF_2\bar{A})e^{2i\Omega T_o}\\ &+2F_1\bar{A}^3e^{i(\Omega-3\omega_o)T_o}+2F_2\bar{A}^3e^{i(2\Omega-3\omega_o)T_o}+NST+c.c.,\end{aligned} \tag{6}$$

where the prime stands for the derivative with respect to T_1, overbar stands for the complex conjugate and NST stands for non-secular terms. Any particular solution of equation (6) contains secular terms and it may contain small-divisor terms depending on the resonance conditions, it can be seen that solutions occur when $2\Omega\cong\omega_0$ and $3\Omega\cong 2\omega_0$. In what follows, we shall investigated super-harmonic oscillation of order two and sub-superharmonic oscillation of order three-to-two of the equation (6).

3. Super-harmonic solution $(2\Omega\cong\omega_o)$

In this case, we study sub-harmonic solution of order two-to-one with introducing the detuning parameters σ_1 to convert the small divisor term into secular terms

$$2\Omega=\omega_0+\varepsilon\sigma_1 \tag{7}$$

and write

$$(\Omega-\omega_o)T_o=\omega_oT_o+\varepsilon\sigma_1T_o=\omega_oT_o+\sigma_1T_1 \tag{8}$$

Inserting equation (8) into equation (6) and eliminating the terms that produce secular terms in u_1 yields the solvability condition

$$2\alpha A-2i\omega_oA'-2i\mu A\omega_o+12\alpha A^2\bar{A}-3\alpha_2A^2\bar{A}+(3A\bar{A}+\frac{1}{2})F_2e^{i\sigma_1T_1}=0 \tag{9}$$

Expressing A in the polar form

$$A(T_1)=\frac{1}{2}a(T_1)e^{i\beta(T_1)} \tag{10}$$

where a and β are real functions of T_1.

Then substituting equation (10) into equation (9) and separating the real and imaginary parts of equation (9), one obtains

$$a' = -a\mu + \frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2 \sin\psi \tag{11}$$

$$a\psi' = a\sigma_1 + \frac{a}{\omega_o}\alpha + \left(\frac{3\alpha}{2\omega_o} - \frac{3\alpha_2}{8\omega_o}\right)a^3 + \frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2 \cos\psi \tag{12}$$

where

$$\psi = \sigma_1 T_1 - \beta \tag{13}$$

It is obvious that, equations (11) and (12) have a trivial solution which of corresponds to the trivial steady state solution. Non-trivial steady state solution correspond to the non-trivial fixed points (equilibrium points) of equations (11) and (12). This means $a' = \psi' = 0$ and are given by

$$\frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2 \sin\psi = a\mu \tag{14}$$

$$\frac{1}{2\omega_o}(1+\frac{3}{2}a^2)F_2 \cos\psi = -\left(\sigma_1 + \frac{\alpha}{\omega_o}\right)a - \left(\frac{3\alpha}{2\omega_o} + \frac{3\alpha_2}{8\omega_o}\right)a^3 \tag{15}$$

Equations (14) and (15) show that there are two possibilities; (trivial solution at $a=0$) and (nontrivial solution at $a \neq 0$). Squaring and adding equations (14) and (15) we get the frequency-response equation

$$\sigma_1 = \frac{-8\alpha\omega_o a^2 - 12\alpha\omega_o a^4 + 3\alpha_2\omega_o a^4 \pm 2\sqrt{4\omega_o^2 a^2 F_2^2 + 12\omega_o^2 a^4 F_2^2 + 9\omega_o^2 a^6 F_2^2 - 16\mu^2\omega_o^4 a^4}}{8\omega_o^2 a^2} \tag{16}$$

Then, the first-order uniform expansion of the solution (first approximation) of equations (1) is given by

$$u = a\cos(2\Omega t - 2\psi) + O(\varepsilon) \tag{17}$$

The analysis of the stability of the trivial solutions is equivalent to the analysis of the linear solutions of equation (9) by neglecting the non-linear terms we get

$$2\alpha A - 2i\omega_o A' - 2i\mu\omega_o A + \frac{1}{2}F_2 e^{i\sigma_1 T_1} = 0 \tag{18}$$

To determine the stability of the trivial steady state solution, it is convenient to rewrite A in the form

$$A = [B(T_1) + ib(T_1)]e^{\frac{i}{2}\sigma_1 T_1} \tag{19}$$

where B and b are separates real and imaginary parts and get

$$b' + \mu b + \Gamma_1 B = 0 \tag{20}$$

$$B' + \mu B - \Gamma_1 b = 0 \tag{21}$$

where $\Gamma_1 = \sigma_1 + \frac{\alpha}{\omega_o}$. Equations (20) and (21) admit solution of the form $(B,b) \propto (B,b)e^{i\theta_o T_1}$, where (B,b) are constants. The eigenvalues of the coefficient matrix of equations (20) and (21) are

$$\theta_o = -\mu \pm i\Gamma_1 \tag{22}$$

Then, the trivial solution is stable if the real parts of both eigenvalues are less than or equal zero.

To determine the stability of the non-trivial steady state solutions given by equations (11) and (12). Let

$$a = a_o + a_1(T_1) \qquad \& \qquad \psi = \psi_o + \psi_1(T_1) \tag{23}$$

where a_0 and ψ_0 correspond to a non-trivial steady state solutions and a_1 and ψ_1 are perturbations which are assumed to be small compared with a_0 and ψ_0. Inserting equation (23) into equations (11) and (12) and linearizing the resulting equations, we obtain

$$a_1' = \mu a_1 - \frac{a_0(8\alpha + 12a_0^2\alpha - 3a_0^2\alpha_2 + 8\sigma_1\omega_0)}{8\omega_0}\psi_1 \tag{24}$$

$$\psi_1' = \frac{(16\alpha + 48a_o^2\alpha + 36a_o^4\alpha - 18a_o^2\alpha_2 - 9a_o^4\alpha_2 + (16-24a_o^2)\sigma_1\omega_o)}{8(2a_o+3a_o^3)\omega_o}a_1 - \frac{(16a_o\mu\omega_o - 24a_o^3\mu\omega_o)}{8(2a_o+3a_o^3)\omega_o}\psi_1 \tag{25}$$

Equations (24) and (25) admit solution of the form $(a_1, \psi_1) \propto (d_1, d_2)e^{\theta T_1}$ where (d_1, d_2) are constants. Provided that

$$\theta = -\frac{2\mu}{c_8} \pm \frac{1}{8}\sqrt{(\frac{1}{c_8^2\omega_o^2}(c_1\alpha^2 + c_2\alpha\alpha_2 + c_3\alpha_2^2 + (c_4\alpha + c_5\alpha_2)\sigma_1\omega_o + (c_6\mu^2 + c_7\sigma_1)^2\omega_o^2))} \tag{26}$$

where

$c_1 = -256 - 1536a^2 - 3456a^4 - 3456a^6 - 1296a^8,\ c_2 = 384a^2 + 1440a^4 + 1728a^6 + 648a^8,$

$c_3 = -108a^4 - 216a^6 - 81a^8,\ c_4 = -512 - 1536a^2 - 1152a^4,\ c_5 = 384a^2 + 576a^4,$

$c_6 = 576a^4,\ c_7 = -256 + 576a^4,\ c_8 = 2 + 3a^2.$

The solution is stable if and only if the real part of each of the eigenvalues of the coefficient of the matrix are less than or equal to zero.

4. Sub-super-harmonic solution $(3\Omega \cong 2\omega_O)$

In this section, we study sub-super-harmonic solution of order three-to-one. To express the nearness of 3Ω to $2\omega_o$, one introduces the detuning parameter σ defined according to

$$3\Omega = 2\omega_o + \varepsilon\sigma \tag{27}$$

Then one can write

$$(3\Omega - 2\omega_o)T_0 = 2\omega_o T_0 + \varepsilon\sigma T_0 = 2\omega_o T_0 + \sigma T_1 \tag{28}$$

Then eliminating the secular terms from equation (6) yields

$$2\alpha A - 2i\omega_o A' - 2i\mu A\omega_o + 12\alpha A^2\bar{A} - 3\alpha_2 A^2\bar{A} + \frac{3}{2}F_2\bar{A}^2 e^{i\sigma_1 T_1} = 0 \tag{29}$$

Using equation (10) into the equation (29) and separating real and imaginary parts, we obtain the following modulation equations

$$a' = -a\mu + \frac{3}{8\omega_o}a^2F_2\sin\gamma \tag{30}$$

$$\frac{1}{3}a\gamma' = (\frac{\sigma}{3} + \frac{\alpha}{\omega_O})a + (\frac{3\alpha}{2\omega_O} - \frac{3\alpha_2}{8\omega_O})a^3 + \frac{3}{8\omega_O}a^2F_2\cos\gamma \tag{31}$$

where $\gamma = \sigma T_1 - 3\beta.$ Substituting a' and γ' equal zero into equations (30) and (31) gives the following equations for the steady state solutions

$$\frac{3}{8\omega_o}a^2F_2\sin\gamma = a\ \mu \tag{32}$$

$$\frac{3}{8\omega_o}a^2F_2\cos\gamma = -\frac{1}{3}a\sigma + \frac{3\alpha_2 a^3}{8\omega_o} - \frac{a\alpha}{\omega_o} - \frac{3\alpha a^3}{2\omega_o} \tag{33}$$

Eliminating the phase angle γ from equations (32) and (33) gives the expression for the solutions curves for the solution $a \neq 0$ as follows

$$(-\frac{1}{3}a\sigma + \frac{3a^3\alpha_2}{8\omega_o} - \frac{a\alpha}{\omega_o} - \frac{3\alpha a^3}{2\omega_o})^2 + a^2\mu^2 - \frac{9}{64\omega_0^2}a^4F_2^2 = 0 \tag{34}$$

i.e.

$$\sigma = \frac{3(-8\alpha\omega_o - 12\alpha\omega_o a^2 + 3\alpha_2\omega_o a^2 \pm \sqrt{9\omega_o^2a^2F_2^2 - 64\mu^2\omega_o^4})}{8\omega_o^2} \tag{35}$$

Now, the analysis of the stability of the trivial solutions is determined as in the preceding section 3, so that we get the eigenvalues equation is similar to equation (22).

Following a procedure similar to that in section 3, one obtains the following eigenvalues that determine the stability of the steady state solutions

$$\theta = -\mu \pm \frac{\sqrt{576\alpha^2 - 1296a^4\alpha^2 + 648a^4\alpha\alpha_2 - 81a^4\alpha_2^2 + 384\alpha\sigma\omega_o + (768\mu^2 + 64\sigma^2)\omega_o^2}}{8\sqrt{3}\omega_0} \quad (36)$$

Consequently, a solution is stable if and only if the real parts of both eigenvalues (36) are less than or equal to zero.

5. Numerical results

In this section, the numerical solution of the frequency response equations (16) and (35) are studied. Frequency response equations (16) and (35) are nonlinear algebraic equations in the amplitude (a). The results are plotted in Figures (1-15), which present the variation of amplitude (a). against the detuning parameters σ_1 and σ.

Figures (1-8) represent the frequency response curves for super harmonic solution of order 2 for the parameters [$\omega_o = 2, \mu = 3, F_2 = \pm 3, \alpha = \pm 1, \alpha_2 = \pm 2$]. In Fig.(1) for positive value of all parameters, we note that the response amplitude has stable single-valued curve and the maximum value exist at the point $\sigma_1 =$ -0.48 . For negative value of some parameters (F_2, α, α_2), we observe that the maximum value shifts to the right so that the maximum value exist at the point $\sigma_1 =$0.57, Fig. (2). When α takes the values 5 and 9, we note that the maximum shift to the left respectively so that the maximum values exist at the points $\sigma_1 =$-2.79 and $\sigma_1 =$-5.06, Fig. (3). For decreasing α with negative values (i.e. α take the value -5 and -9), we observe that the maximum shift to the right respectively so that the maximum values exist at the points $\sigma_1 = 2.79$ and $\sigma_1 = 5.05$, Fig.(4). When $\alpha_2 = 13$, we note that the singled-valued curves are intersect at the the same maximum value, see Fig. (5). For increasing and decreasing the coefficient of nonlinear external excitation F_2 respectively, we observe that the singled-valued curves shift upward and downward respectively and have increasing and decreasing maximum values, Fig. (6,7,8).

Figures (9-15) represent the frequency response curves for sub-super harmonic solution of order ($\frac{3}{2}$) for the parameters [$\omega_o = 0.3,\ \mu = 0.2,\ F_2 = 3,\ \alpha = 0.01,\ \alpha_2 = 2$].

In Fig.(9) for positive values, we observe that the response amplitude has multivalued curve which consistes of two branches while the lower branch has unstable solutions and the upper branch has stable solutions and there exist a saddle nodes bifurcations at the points $\sigma =$-4.28 and σ = -4.34. When α_2 takes the values 2 and 5, we observe that the multivalued curve contracted so that the upper and lower branches are shifts to downward so that these branches have decreased magnitudes respectively. The saddle nodes bifurcations exist at the points σ = -4.30 and $\sigma =$-1.69, Fig. (10). For decreasing α_2 with negative values (i.e. α_2 takes the values -2, -5), we note that the multivalued curve is contracted so that the upper and lower branches have decreased magnitudes respectively and the saddle nodes bifurcations exist at the points $\sigma =$4.01 and σ = 1.46, Fig.(11). As the parameter α is decreased with positive values (i.e. α takes the values 0.1 and 0.01), we get the same variation as in Fig. (10) so that the saddle nodes bifurcations exist at the points $\sigma =$-6.23 and $\sigma =$-4.33, Fig. (12). When the coefficient of nonlinear external excitation F_2 is decreased, we observe that the multivalued curve is contracted so that the upper and lower branches are shifts to downward and upward so that the upper branch has decreased magnitudes and the lower branche has increased magnitudes. As $F_2 = 0.3$, we observe that the multivalued curve is contracted and given semi-oval and the saddle nodes bifurcations exist at the points $\sigma = 2.01$ and $\sigma = 2.11$, Fig. (13). For increasing the damping factor μ, we note that the multivalued curve is contracted and the saddle nodes bifurcations exist at the points $\sigma =$-3.79and σ = 3.89, Fig.(14). When the natural frequency ω_o takes the values 0.9 and 2, we observe that the multivalued curve is contracted respectively so that the upper branch has stable and unstable solutions while the lower branch has stable and unstable solutions and these branches are intersect at the point $\sigma =$-4.31. The saddle nodes bifurcations exist at the points $\sigma =$-1.40 and $\sigma =$-0.49, Fig. (15).

6. Summary and conclusions

An analytical and numerical technique is used to predict the qualitative change taking place in the stable solutions of the non-linear modified Duffing's equation subjected to a bi-harmonic parametric and external excitations. The multiple time scales are used to investigate first-order approximate analytical solution. The modulation equations (reduced equations) of the amplitude and phase are obtained. Steady state solutions and their stability are determined. The following conclusions can be deduced from the analysis:

From the frequency-response curves of super-harmonic oscillation of order two in Figs. (1-8), we note that the response amplitude has single-valued curve and all solutions are stable. The maximum value shifts to the left and right for increasing and decreasing with decreasing α with negative values respectively. The maximum value shifts upward for increasing F_2, ω_o and for decreasing μ. The maximum value shifts downward for decreasing F_2 and for increasing ω_o and μ.

From the frequency-response curves of sub-superharmonic oscillation of order ($\frac{3}{2}$), we observe that the response amplitude has multivalued curve. The stable and unstable solutions are exist in the upper and lower branches respectively. For positive (negative) values, we note that the multivalued curve bents to the right (left) and has harding (softing) behavior. When $F_2 = 0.3$ and $\mu = 4$, we observe that the multivalued curve contracted and given semi-ovals. The upper branch of the multivalued curves are intersect at the same point $\sigma =$-4.31 when ω_o takes the values 0.3, 0.9 and 2.

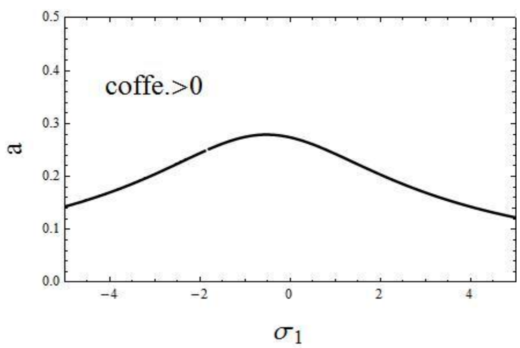

Fig 1.

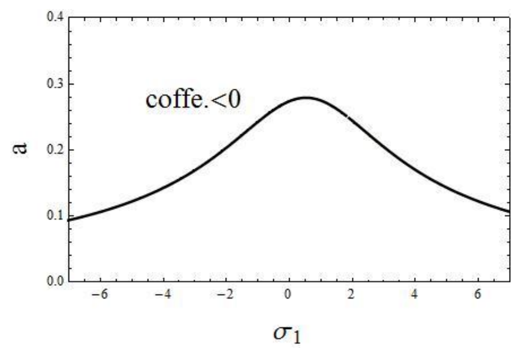

Fig 2.

The frequency response curves of the super-harmonic solution of order 2 for the parameters $\omega_o = 2, \mu = 3, F_2 = \pm 3, \alpha = \pm 1, \alpha_2 = \pm 2$.

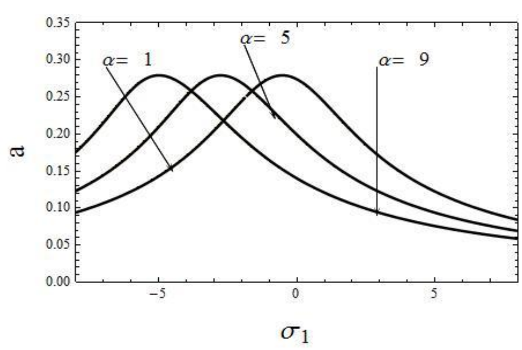

Fig 3.

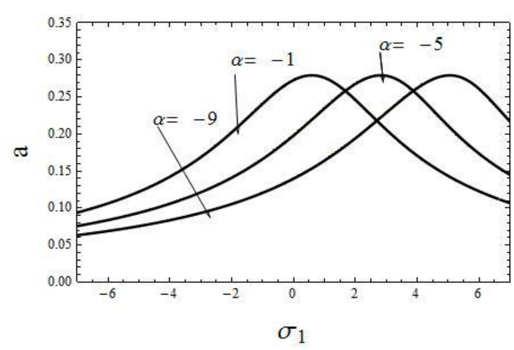

Fig 4.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α .

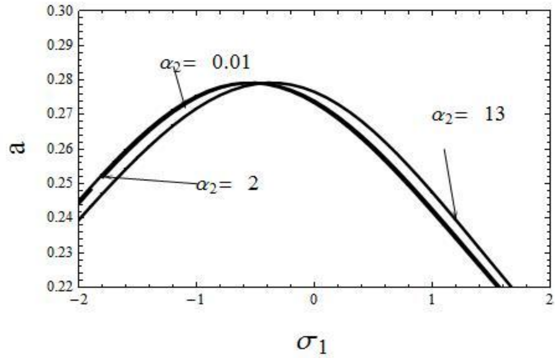

Fig 5.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2

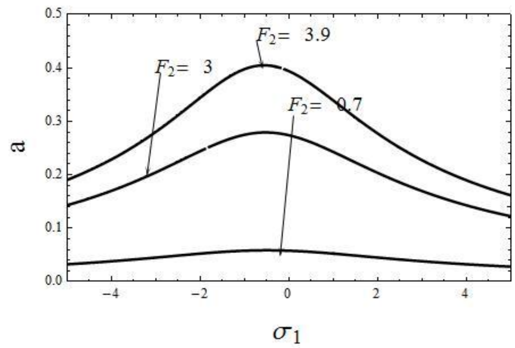

Fig 6.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2

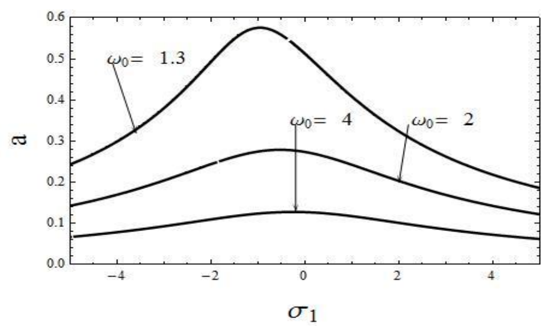

Fig 7.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

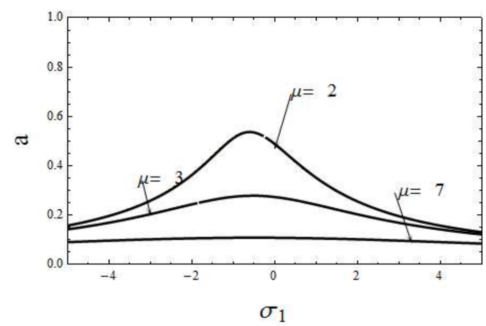

Fig 8.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing μ

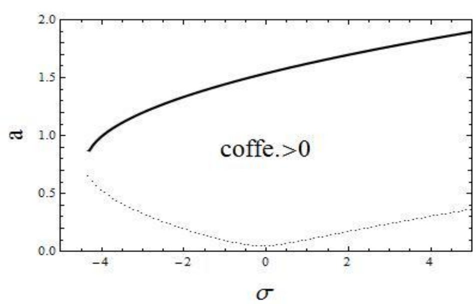

Fig 9.

The frequency response curves of the sub-super-harmonic solution of order $\frac{3}{2}$ for the parameters

$$\omega_o = .3, \mu = 0.2, F_2 = 3, \alpha = .01, \alpha_2 = 2.$$

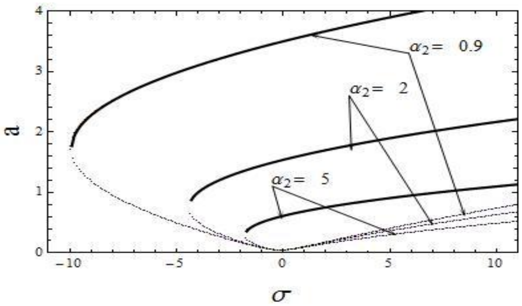

Fig 10.

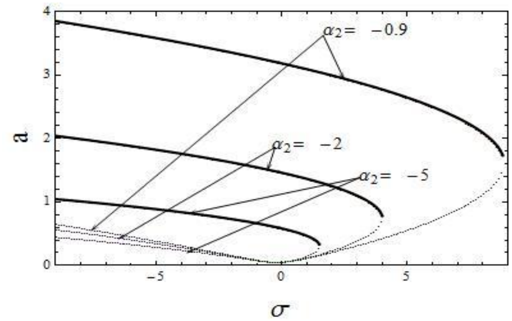

Fig 11.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α_2 .

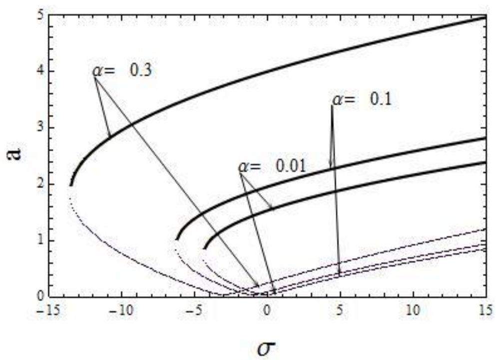

Fig 12.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing α

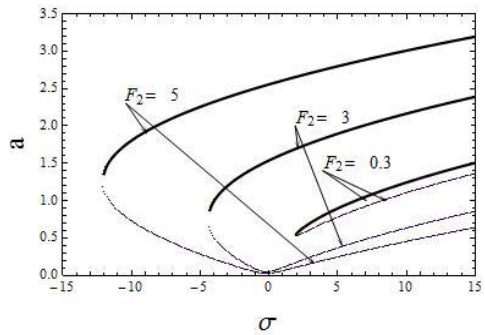

Fig 13.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing F_2

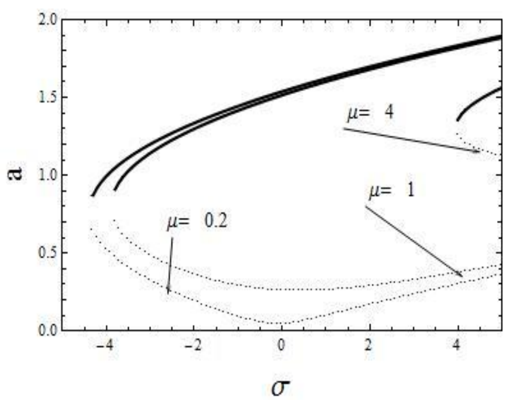

Fig 14.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing ω_o

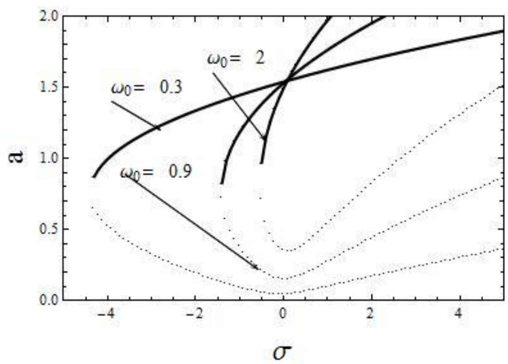

Fig 15.

Variation of the amplitude of the response with the detuning parameter for increasing and decreasing μ

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